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On the evaluation formula for Jack polynomials with prescribed symmetry

P J Forrester, D S McAnally and Y Nikoyalevsky

Department of Mathematics and Statistics, University of Melbourne, Victoria 3010, Australia

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Abstract

The Jack polynomials with prescribed symmetry are obtained from the nonsymmetric polynomials via the operations of symmetrization, antisymmetrization and normalization. After dividing out the corresponding antisymmetric polynomial of smallest degree, a symmetric polynomial results. Of interest in applications is the value of the latter polynomial when all the variables are set equal. Dunkl has obtained this evaluation, making use of a certain skew-symmetric operator. We introduce a simpler operator for this purpose, thereby obtaining a new derivation of the evaluation formula. An expansion formula of a certain product in terms of Jack polynomials with prescribed symmetry implied by the evaluation formula is used to derive a generalization of a constant term identity due to Macdonald, Kadell and Kaneko. Although we do not give the details in this paper, the operator introduced here can be defined for any reduced crystallographic root system, and used to provide an evaluation formula for the corresponding Heckman–Opdam polynomials with prescribed symmetry.

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1. Introduction

The type-A periodic Calogero–Sutherland quantum many-body system with exchange terms is described by the Schrödinger operator

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \left(\frac{\pi}{L}\right)^2 \sum_{1 \leqslant j < k \leqslant N} \frac{(1/\alpha)((1/\alpha) - s_{jk})}{\sin^2(\pi(x_k - x_j)/L)}. \tag{1.1}$$

In (1.1) all particles are confined to the interval $[0, L]$ with periodic boundary conditions, and s_{jk} is the exchange operator which permutes the variables x_j and x_k. It is a standard result that for $0 < x_1 < x_2 < \cdots < x_N < L$ the ground state for (1.1) is

$$\psi_0 := \prod_{1 \leqslant j < k \leqslant N} \left(\sin(\pi(x_k - x_j)/L)\right)^{1/\alpha} \tag{1.2}$$

and it is similarly a standard result that with $z_j := \mathrm{e}^{2\pi \mathrm{i} x_j/L}$ the excited states have the form

$$\psi_0\left(\prod_{j=1}^N z_j^l\right) f(z_1,\ldots,z_N) \tag{1.3}$$

where $l \in \mathbb{Z}_{\leqslant 0}$ and f is a homogeneous multivariable polynomial. The polynomial f is a type of Jack polynomial. In its most basic form the Jack polynomial is nonsymmetric [12], however the transposition operator $s_i := s_{i\,i+1}$ commutes with H so we are free to restrict the eigenfunctions to subspaces symmetric or antisymmetric with respect to certain sets of coordinates. In physical terms this corresponds to having a multicomponent system consisting of a mixture of bosons and fermions. The polynomial part of the eigenfunction is then referred to as a Jack polynomial with prescribed symmetry [2].

The Jack polynomials with prescribed symmetry are to be denoted $S_{\eta^*}^{(I,J)}(z)$, requiring the two labels η^* and (I, J) for their unique specification. The set I in the label (I, J) determines the variables in which $S_{\eta^*}^{(I,J)}$ is symmetric, while the set J determines the variables in which $S_{\eta^*}^{(I,J)}$ is antisymmetric. Explicitly

$$s_i S_{\eta^*}^{(I,J)}(z) = S_{\eta^*}^{(I,J)}(z) \quad (i \in I) \qquad s_j S_{\eta^*}^{(I,J)}(z) = -S_{\eta^*}^{(I,J)}(z) \quad (j \in J) \tag{1.4}$$

and thus

$$S_{\eta^*}^{(I,J)}(z) \propto \mathrm{Sym}_I \mathrm{Asym}_J\, E_\eta(z). \tag{1.5}$$

For (1.4) to make sense we must have that I and J are disjoint subsets of $\{1, 2, \ldots, N-1\}$ such that

$$i-1, i+1 \notin J \qquad \text{for} \quad i \notin I \qquad \text{and} \qquad j-1, j+1 \notin I \qquad \text{for} \quad j \in J \tag{1.6}$$

(thus, representing the members of I by red dots and the members of J by black dots on the lattice $\{1, 2, \ldots, N-1\}$, there are no consecutive lattice points marked with different coloured dots). The index η^* is a composition such that

$$\eta_i^* \geqslant \eta_{i+1}^* \quad \forall i \in I \qquad \eta_j^* > \eta_{j+1}^* \quad \forall j \in J. \tag{1.7}$$

Let the set J be decomposed as a union of sets of consecutive integers, denoted J_s say, and let $\tilde{J}_s := J_s \cup \{\max(J_s)+1\}$ and $\tilde{J} := \cup_s \tilde{J}_s$. Then from (1.4) the polynomial $S_{\eta^*}^{(I,J)}$ can be factorized in the form

$$S_{\eta^*}^{(I,J)}(z) = \prod_s \Delta_{\tilde{J}_s}(z)\, U_{\eta^*}^{(I,J)}(z) \qquad \Delta_X(z) := \prod_{\substack{i<i' \\ (i,i')\in X}} (z_i - z_{i'}) \tag{1.8}$$

where $U_{\eta^*}^{(I,J)}(z)$ is symmetric with respect to s_i for $i \in I \cup J$. Our interest is in the evaluation of $U_{\eta^*}^{(I,J)}(1^N)$, where

$$1^N := (z_1,\ldots,z_N)\big|_{z_j=1\,(j=1,\ldots,N)}.$$

In fact, the value of the closely related quantity

$$\left.\frac{\mathrm{Sym}_I \mathrm{Asym}_J\, E_\eta(z)}{\prod_s \Delta_{\tilde{J}_s}(z)}\right|_{z=1^N}$$

is already known from the work of Dunkl [4]. Recalling (1.5) we see the value of $U_{\eta^*}^{(I,J)}(z)$ follows once the proportionality constant in the former is determined. In proposition 2.2 we determine the proportionality constant. However, we will not then be done with the problem. Rather, we seek a self-contained derivation, motivated by our desire to obtain the evaluation formula for the analogue of $U_{\eta^*}^{(I,J)}(1^N)$ in the case of Heckman–Opdam polynomials with prescribed symmetry. On this point, we recall (see, for example, [10, 11]) that the Jack polynomials are the special case of the Heckman–Opdam polynomials corresponding to the type-A root system; the latter can be constructed for all reduced crystallographic root systems and are also the polynomial part of certain Schrödinger operators. The derivation given here does indeed permit a generalization to this more general setting, although we reserve the details until a later publication.

By way of further motivation, we point out that knowledge of the evaluation of $U_{\eta^*}^{(I,J)}(1^N)$ has been of essential use in the exact computation of retarded Green functions for spin generalizations of the Hamiltonian (1.1) [7]. The reason is that the value of $U_{\eta^*}^{(I,J)}(1^N)$ combined with the Cauchy product expansion involving the $S_{\eta^*}^{(I,J)}$ gives the expansion of the product

$$\prod_{i\notin\tilde{J}}(1-z_i)^{r-1}\prod_s\prod_{j\in\tilde{J}_s}(1-z_j)^{r-|J_s|} \tag{1.9}$$

in terms of $\{S_{\eta^*}^{(I,J)}\}$, which is one of the main technical requirements in the computation of the retarded Green functions. The expansion of (1.9) and a corresponding constant term (CT) identity will be discussed in section 4.

We begin in section 2 by revising essential properties of the Jack polynomials with prescribed symmetry. In section 3 we introduce an operator O_J which transforms $S_{\eta^*}^{(I,J)}$ to be proportional to $S_{\eta^*}^{(I\cup J,\emptyset)}$. We will show that the same operator acting on the right-hand side of the first equation in (1.8), followed by evaluation at $z=1^N$, acts only on $\prod_s \Delta_{\tilde{J}_s}(z)$. By evaluating $O_J(\prod_s \Delta_{\tilde{J}_s}(z))$ the value of $U_{\eta^*}^{(I,J)}(1^N)$ follows. This is stated in proposition 3.6. In section 4 we contrast our method with that of Dunkl. We also revise the expansion of (1.9) in terms of $\{S_{\eta^*}^{(I,J)}\}$, obtained from knowledge of $U_{\eta^*}^{(I,J)}(1^N)$. This is then used to obtain a certain CT identity involving $S_{\eta^*}^{(I,J)}(z)$.

2. Properties of the polynomials $S_{\eta^*}^{(I,J)}$

2.1. Definitions and preliminaries

Let s_{jk} denote the permutation operator which when acting on functions $f=f(z_1,\ldots,z_N)$ interchanges z_j and z_k. In the notation of Knop and Sahi [8] we introduce the type-A Cherednik operators by

$$\xi_i=\alpha z_i\frac{\partial}{\partial z_i}+\sum_{p<i}\frac{z_i}{z_i-z_p}(1-s_{ip})+\sum_{p>i}\frac{z_p}{z_i-z_p}(1-s_{ip})+1-i \qquad i=1,\ldots,N. \tag{2.1}$$

The relation to the type-A root system becomes apparent by comparing (2.1) with the general Cherednik operator associated with a root system,

$$D_{\vec{\lambda}}:=\sum_{j=1}^N\lambda_j z_j\frac{\partial}{\partial z_j}+\sum_{\vec{\beta}\in R_+}k_{\vec{\beta}}\frac{\langle\vec{\lambda},\vec{\beta}\rangle}{z^{\vec{\beta}}-1}(1-s_{\vec{\beta}})+\langle\vec{\lambda},\vec{\rho}_k\rangle \tag{2.2}$$

where $\vec{\lambda}=(\lambda_1,\ldots,\lambda_N)$ denotes a real vector with N components, $\langle\,,\,\rangle$ denotes the dot product and

$$z^{\vec{\beta}}:=z_1^{\beta_1}z_2^{\beta_2}\cdots z_N^{\beta_N}. \tag{2.3}$$

The quantities R_+, $k_{\vec{\beta}}$, $s_{\vec{\beta}}$ and $\vec{\rho}_k$ depend on the root system R. In the type-A case, with $\vec{e}_j$ denoting the elementary unit vector with component j equal to 1,

$$R_+ = \{\vec{e}_j - \vec{e}_k | 1 \leqslant j < k \leqslant N\} \qquad k_{\vec{\beta}} = 1/\alpha \tag{2.4}$$

$$s_{\vec{\beta}} = s_{jk} \qquad \text{for} \quad \vec{\beta} = \vec{e}_j - \vec{e}_k \qquad \vec{\rho}_k = \sum_{j=1}^{N} \frac{1}{\alpha}\left(\frac{N+1}{2} - j\right)\vec{e}_j. \tag{2.5}$$

Using these explicit formulae in (2.1), a short calculation shows that in the type-A case

$$D_{\vec{\lambda}} = \frac{1}{\alpha}\sum_{j=1}^{N} \lambda_j \left(\xi_j + \frac{N-1}{2}\right). \tag{2.6}$$

In particular, with $\vec{\beta}$ as specified in (2.5),

$$D_{\vec{\beta}} = \frac{1}{\alpha}(\xi_j - \xi_k). \tag{2.7}$$

A fundamental property of $\{\xi_i\}$ is that they form a commuting family of operators which permit a set of simultaneous polynomial eigenfunctions, labelled by a composition $\eta := (\eta_1, \eta_2, \ldots, \eta_N)$, $\eta_j \geqslant 0$, and homogeneous of degree $|\eta| := \sum_{j=1}^{N} \eta_j$. These are the nonsymmetric Jack polynomials $E_\eta(z;\alpha) =: E_\eta(z)$, which are uniquely characterized as the solution of the eigenvalue equation

$$\xi_i E_\eta = \bar{\eta}_i E_\eta \qquad \bar{\eta}_i := \alpha\eta_i - \#\{k < i | \eta_k \geqslant \eta_i\} - \#\{k > i | \eta_k > \eta_i\} \tag{2.8}$$

possessing a special triangularity structure when expanded in terms of monomials. To specify this structure, let us denote by η^+ the partition corresponding to the composition η. Let η and ν, $\eta \neq \nu$ be compositions such that $|\eta| = |\nu|$, and introduce the partial ordering $<$, known as the dominance ordering, by the statement $\nu < \eta$ if $\sum_{j=1}^{p} \nu_j < \sum_{j=1}^{p} \eta_j$ for each $p = 1, \ldots, N$. A further partial ordering $\prec$ is defined on compositions by the statement that $\nu \prec \eta$ if $\nu^+ < \eta^+$, or in the case $\nu^+ = \eta^+$, if $\nu < \eta$. In terms of the monomials ordered by the partial ordering $\prec$, the non-symmetric Jack polynomials have the expansion

$$E_\eta(z) = z^\eta + \sum_{\nu \prec \eta} c_{\eta\nu} z^\nu \tag{2.9}$$

for some coefficients $c_{\eta\nu} = c_{\eta\nu}(\alpha)$.

Contained in the set of positive roots R_+ (recall (2.4)) are the so-called simple roots

$$\vec{\alpha}_j := \vec{e}_j - \vec{e}_{j+1} \qquad j \in \{1, 2, \ldots, N-1\} \tag{2.10}$$

which form a basis of R_+. We consider two subsets $I, J \subseteq \{1, 2, \ldots, N-1\}$ such that $I \cap J = \emptyset$, and write $W_{I\cup J} := \langle s_j | j \in I \cup J\rangle$ where $s_j := s_{\vec{\alpha}_j} = s_{j\,j+1}$. We define the operation $\mathcal{O}_{I,J}$ on monomials by

$$\mathcal{O}_{I,J}(z^\lambda) = \sum_{w \in W_{I\cup J}} \det{}_J(w) z^{w(\lambda)}$$

where $w(\lambda) := (\lambda_{w^{-1}(1)}, \ldots, \lambda_{w^{-1}(N)})$, and $\det_J(w) = 1$ if a decomposition of w into a product of transpositions has an even number of factors from $\{s_j | j \in J\}$ and $\det_J(w) = -1$ if such a decomposition has an odd number of entries from $\{s_j | j \in J\}$. This operator is extended to general analytic functions by linearity.

Let us suppose that in addition to requiring $I \cap J = \emptyset$, we also have the condition (1.6). In this circumstance $W_{I\cup J} = W_I W_J = W_J W_I$ and furthermore

$$\begin{aligned} &\mathcal{O}_{I,J} = \mathcal{O}_I \mathcal{O}_J = \mathcal{O}_J \mathcal{O}_I \qquad \text{with} \quad \mathcal{O}_I = \text{Sym}_I = \prod_s \text{Sym}_{I_s} \\ &\mathcal{O}_J = \text{Asym}_J = \prod_s \text{Asym}_{J_s} \end{aligned} \tag{2.11}$$

where $I = \cup_s I_s$ and $J = \cup_s J_s$ (recall the beginning of the paragraph below (1.4)).

Our interest is in the polynomials $\mathcal{O}_{I,J} E_\eta(z)$. From the definition of $\mathcal{O}_{I,J}$ we see that

$$\mathcal{O}_{I,J} E_\eta(z) = 0 \qquad \text{if} \quad \eta_j = \eta_{j'} \qquad \text{for any} \quad j \neq j' \in \tilde{J}_s, \text{ some } s \tag{2.12}$$

where $\tilde{J}_s$ is defined as in the beginning of the paragraph below (1.7). On the other hand, if $\eta_j \neq \eta_{j'}$ for all $j \neq j' \in \tilde{J}_s$, all s, then

$$\mathcal{O}_{I,J} E_\eta(z) = a_\eta^{(I,J)} S_{\eta^*}^{(I,J)}(z) \qquad S_{\eta^*}^{(I,J)}(z) = z^{\eta^*} + \sum_{\nu \prec \eta^*} \tilde{c}_{\eta\nu} z^\nu \tag{2.13}$$

for some $a_\eta^{(I,J)}$ and $\tilde{c}_{\eta\nu} = \tilde{c}_{\eta\nu}(\alpha)$. In (2.13) η^* is the unique element of $W_{I\cup J}(\eta)$ such that (1.7) holds. The polynomials $S_{\eta^*}^{(I,J)}$, which were first introduced in [2], and subsequently studied in [1, 4, 7], are referred to as the Jack polynomials with prescribed symmetry. They have the symmetry properties (1.4) which, together with the structural formula in (2.13), can be used to uniquely characterize the polynomials.

Our subsequent calculations require the explicit value of the proportionality constant $a_\eta^{(I,J)}$ in (2.13). One way of obtaining this is to make use of the explicit expansion of $S_{\eta^*}^{(I,J)}(z)$ in terms of $\{E_\nu\}$. Let us first revise the derivation of this latter expansion [1]. Now, we know [12] that the action of s_i on E_η is given by

$$s_i E_\eta(z) = \begin{cases} \dfrac{1}{\bar{\delta}_{i,\eta}} E_\eta(z) + \left(1 - \dfrac{1}{\bar{\delta}_{i,\eta}^2}\right) E_{s_i\eta}(z) & \eta_i > \eta_{i+1} \\ E_\eta(z) & \eta_i = \eta_{i+1} \\ \dfrac{1}{\bar{\delta}_{i,\eta}} E_\eta(z) + E_{s_i\eta}(z) & \eta_i < \eta_{i+1} \end{cases} \tag{2.14}$$

where $\bar{\delta}_{i,\eta} := \bar{\eta}_i - \bar{\eta}_{i+1}$, with $\bar{\eta}_i$ specified by (2.6). It follows from (2.14), (2.11) and (2.13) that

$$S_{\eta^*}^{(I,J)}(z) = \sum_{\mu \in W_{I\cup J}(\eta^*)} \hat{c}_{\eta^*\mu} E_\mu(z) \qquad \hat{c}_{\eta^*\eta^*} = 1. \tag{2.15}$$

Moreover, the coefficients $\hat{c}_{\eta^*\mu}$ in (2.15) can be computed explicitly in terms of the quantities

$$d_\eta := \prod_{(i,j)\in\eta} \big(\alpha(a(i,j)+1) + l(i,j) + 1\big) \qquad d'_\eta := \prod_{(i,j)\in\eta} \big(\alpha(a(i,j)+1) + l(i,j)\big) \tag{2.16}$$

where the notation $(i, j) \in \eta$ refers to the diagram of the composition η, while

$$a(i,j) := \eta_i - j \qquad l(i,j) := \#\{k < i | j \leqslant \eta_k + 1 \leqslant \eta_i\} + \#\{k > i | j \leqslant \eta_k \leqslant \eta_i\}. \tag{2.17}$$

Our derivation makes use of the fact that the quantities d_η and $d_{\eta'}$ have the properties [13]

$$\frac{d_{s_i\eta}}{d_\eta} = \frac{\bar{\delta}_{i,\eta} + 1}{\bar{\delta}_{i,\eta}} \qquad \frac{d'_{s_i\eta}}{d'_\eta} = \frac{\bar{\delta}_{i,\eta}}{\bar{\delta}_{i,\eta} - 1} \qquad \eta_i > \eta_{i+1}. \tag{2.18}$$

Proposition 2.1. *Let $w \in W_{I\cup J}$ be decomposed as $w = w_I w_J$ where $w_I \in W_I$, $w_J \in W_J$. Let $w\eta^* = \mu$ and $w_I\eta^* = \mu_I$. Then the coefficents in (2.15) are specified by*

$$\hat{c}_{\eta^*\mu} = \det{}_J(w) \frac{d'_{\eta^*} d_\mu}{d'_{\mu_I} d_{\mu_I}}. \tag{2.19}$$

Proof. We write

$$\sum_{\mu \in W_{I\cup J}(\eta^*)} \hat{c}_{\eta^*\mu} E_\mu(z) = \sum_{\substack{\mu \in W_{I\cup J}(\eta^*) \\ \mu_i \leqslant \mu_{i+1}}} \chi_{\mu_i\mu_{i+1}} \big(\hat{c}_{\eta^*\mu} E_\mu(z) + \hat{c}_{\eta^* s_i\mu} E_{s_i\mu}(z)\big) \tag{2.20}$$

where $\chi_{\mu_i\mu_{i+1}} = \frac{1}{2}$ for $\mu_i = \mu_{i+1}$, $\chi_{\mu_i\mu_{i+1}} = 1$ otherwise. Applying the operator s_i for $i \in I$ and s_j for $j \in J$ to (2.20) with $\hat{c}_{\eta^*\mu}$ given by (2.19), we see by making use of (2.18) that the properties (1.4) hold. Also, with $w_I = w_J = \mathrm{Id}$ (the identity), and thus $\mu = \mu_I = \mu_J = \eta^*$ and $\det_J(w) = 1$, we see that $\hat{c}_{\eta^*\eta^*} = 1$. □

Proposition 2.1 can be used to determine $a_\eta^{(I,J)}$. To present the result requires some notation. Let

$$\mathcal{M}_{I,\eta} = \#\{\sigma' \in W_I | \sigma'(\eta) = \eta\}$$

and write $\eta^{(\epsilon_I,\epsilon_J)}$, where $\epsilon_I, \epsilon_J \in \{+, 0, -\}$ to denote the element of $W_{I\cup J}(\eta)$ with the property that $\eta^{(+,\cdot)}$ ($\eta^{(\cdot,+)}$) has $\eta_i^{(+,\cdot)} \geqslant \eta_{i+1}^{(+,\cdot)}$ for all $i \in I$ ($\eta_j^{(\cdot,+)} > \eta_{j+1}^{(\cdot,+)}$ for all $j \in J$), $\eta^{(-,\cdot)}$ ($\eta^{(\cdot,-)}$) has $\eta_i^{(-,\cdot)} \leqslant \eta_{i+1}^{(-,\cdot)}$ for all $i \in I$ ($\eta_j^{(\cdot,-)} < \eta_{j+1}^{(\cdot,-)}$ for all $j \in J$), while $\eta^{(0,\cdot)}$ ($\eta^{(\cdot,0)}$) has $\eta_i^{(0,\cdot)} = \eta_i$, $\eta_{i+1}^{(0,\cdot)} = \eta_{i+1}$ for all $i \in I$ ($\eta_j^{(\cdot,0)} = \eta_j$, $\eta_{j+1}^{(\cdot,0)} = \eta_{j+1}$ for all $j \in J$).

Proposition 2.2. *With $a_\eta^{(I,J)}$ defined by (2.13) we have*

$$a_\eta^{(I,J)} = \det_J(w_J)\mathcal{M}_{I,\eta}\frac{d'_\eta d'_{\eta^{(-,+)}} d_{\eta^{(-,+)}}}{d'_{\eta^{(0,+)}} d_{\eta^{(0,+)}} d'_{\eta^{(-,-)}}} \tag{2.21}$$

where w_J is such that $w_J\eta = \eta^{(0,+)}$.

Proof. We model our proof on the derivation given in [2] of the value of $a_\eta^{(I,J)}$ in the case that $I = \{1, \ldots, N-1\}$ and $J = \emptyset$ (symmetrization in all variables). The first step is to introduce the polynomial

$$G(x, y) = \sum_{\mu\in W_{I\cup J}(\eta^*)} \frac{d_\mu}{d'_\mu} E_\mu(x)E_\mu(y) \tag{2.22}$$

which from (2.14) is seen to have the property

$$s_i^{(x)}G(x, y) = s_i^{(y)}G(x, y) \qquad \text{for} \quad i \in I \cup J.$$

In particular

$$\mathcal{O}^{(x)}G(x, y) = \mathcal{O}^{(y)}G(x, y). \tag{2.23}$$

Substituting (2.22) in (2.23) and recalling (2.13) shows

$$S_{\eta^*}^{(I,J)}(x) \sum_{\mu\in W_{I\cup J}(\eta^*)} \frac{d_\mu}{d'_\mu} a_\mu^{(I,J)} E_\mu(y) = S_{\eta^*}^{(I,J)}(y) \sum_{\mu\in W_{I\cup J}(\eta^*)} \frac{d_\mu}{d'_\mu} a_\mu^{(I,J)} E_\mu(x).$$

It follows from this that

$$S_{\eta^*}^{(I,J)}(x) = a_{\eta^*} \sum_{\mu\in W_{I\cup J}(\eta^*)} \frac{d_\mu}{d'_\mu} a_\mu^{(I,J)} E_\mu(x) \tag{2.24}$$

for some constant a_{η^*}. Comparing (2.24) with (2.15) and recalling the result of proposition 2.1 we thus have

$$a_{\eta^*}\frac{d_\mu}{d'_\mu} a_\mu^{(I,J)} = \det_J(w_J)\frac{d'_{\eta^*} d_\mu}{d'_{\mu_I} d_{\mu_I}} \tag{2.25}$$

with $\mu_I = \mu^{(+,0)}$ and w_J such that $w_J\mu = \mu^{(0,+)}$. The identity (2.25) must hold for all $\mu \in W_{I\cup J}(\eta^*)$. In particular it must hold for the smallest composition with respect to the partial ordering $\prec$, namely $\eta^{(-,-)}$. This composition is special in that

$$\mathcal{O}_{I,J} E_{\eta^{(-,-)}}(z) = \det_J(w_{J'})\mathcal{M}_{I,\eta} S_{\eta^*}^{(I,J)}(z)$$

where $w_{J'}\eta^{(-,-)} = \eta^{(-,+)}$. Thus

$$a^{(I,J)}_{\eta^{(-,-)}} = \det_J(w_{J'})\mathcal{M}_{I,\eta}$$

which when substituted in (2.25) with $\mu = \eta^{(-,-)}$ implies

$$a_{\eta^*} = \frac{1}{\mathcal{M}_{I,\mu}} \frac{d'_{\eta^*} d'_{\eta^{(-,-)}}}{d'_{\eta^{(-,+)}} d_{\eta^{(-,+)}}}. \tag{2.26}$$

Substituting (2.26) in (2.25) with $\mu = \eta$ (of course $\eta \in W_{I\cup J}(\eta^*)$) gives (2.21). □

In our derivation of the evaluation formula for the polynomial $U^{(I,J)}_{\eta^*}(z)$ in (1.8), it is a corollary of proposition 2.2 giving the evaluation formula for $S^{(I,\emptyset)}_{\eta^*}$ which is of use.

Corollary 2.3. *For the Jack polynomial with prescribed symmetry, $S^{(I,\emptyset)}_{\eta^*}$, constructed out of symmetrization operations only, we have the evaluation formula*

$$S^{(I,\emptyset)}_{\eta^*}(1^N) = \frac{|\mathrm{Sym}\, I|}{a^{(I,\emptyset)}_{\eta^*}} E_{\eta^*}(1^N) = \frac{|\mathrm{Sym}\, I|}{\mathcal{M}_{I,\eta^*}} \frac{e_{\eta^+}}{d_{\eta^{*(-,\cdot)}}} \tag{2.27}$$

where η^+ denotes the unique partition which can be formed from η^* and

$$e_{\eta^+} := \alpha^{|\eta|}[1 + N/\alpha]^{(\alpha)}_{\eta^+} \qquad [u]^{(\alpha)}_{\eta^+} := \prod_{j=1}^{N} \frac{\Gamma(u - \frac{1}{\alpha}(j-1) + \eta^+_j)}{\Gamma(u - \frac{1}{\alpha}(j-1))}. \tag{2.28}$$

Proof. The first equality follows immediately from (2.13) with $J = \emptyset$, while the second equality follows from proposition 2.2 with $J = \emptyset$ and $\eta = \eta^*$, which gives

$$a^{(I,\emptyset)}_{\eta} = \mathcal{M}_{I,\eta} \frac{d_{\eta^{(-,\cdot)}}}{d_\eta}$$

and the well known result [13]

$$E_\eta(1^N) = \frac{e_{\eta^+}}{d_\eta}. \tag{2.29}$$

□

3. Evaluation formula for $U^{(I,J)}_{\eta^*}$

3.1. A special operator

Of central importance to our eventual evaluation of $U^{(I,J)}_{\eta^*}$ is the operator

$$O_J := \prod_{\vec{\beta} \in R_{J,+}} \left(D_{\vec{\beta}} + \frac{1}{\alpha} \right) \tag{3.1}$$

where

$$R_{J,+} = \{\vec{e}_j - \vec{e}_k | 1 \leqslant j < k \leqslant N,\ j,k \in \tilde{J}_s,\ \text{some } s\}$$

and $D_{\vec{\beta}}$ is specified in terms of the ξ_i by (2.6) (in the case $R_{J,+} = R_+$ this operator is introduced for general reduced crystallographic root systems in [11]). Also, $R_{J,-}$ is defined by

$$R_{J,-} = \{-\vec{e}_j + \vec{e}_k | 1 \leqslant j < k \leqslant N,\ j,k \in \tilde{J}_s,\ \text{some } s\}.$$

We seek an algebraic relation for $s_i O_J$. First, one can check from the relations

$$\xi_i s_i - s_i \xi_{i+1} = 1 \qquad \xi_{i+1} s_i - s_i \xi_i = -1 \qquad [\xi_i, s_j] = 0 \qquad (j \neq i-1, i)$$

(the subalgebra of the degenerate type-A Hecke algebra satisfied by $\{\xi_i, s_j\}$) and (2.7), or alternatively directly from (2.2), that for $\vec{\beta}_i = \vec{e}_i - \vec{e}_{i+1}$,

$$s_i D_{\vec{\beta}} - D_{s_i(\vec{\beta})} s_i = \frac{1}{\alpha}\langle \vec{\beta}, \vec{\beta}_i\rangle. \tag{3.2}$$

Now we rewrite (3.1) to read

$$O_J = \prod_{\substack{\vec{\beta}\in R_{J,+}\\ \langle\vec{\beta},\vec{\beta}_i\rangle=0}} \left(D_{\vec{\beta}} + \frac{1}{\alpha}\right) \prod_{\substack{\vec{\beta}\in R_{J,+}\\ \langle\vec{\beta},\vec{\beta}_i\rangle\neq 0,\ \vec{\beta}\neq\vec{\beta}_i}} \left(D_{\vec{\beta}} + \frac{1}{\alpha}\right) \begin{cases} 1 & \vec{\beta}_i \notin R_{J,+} \\ \left(D_{\vec{\beta}_i} + \frac{1}{\alpha}\right) & \vec{\beta}_i \in R_{J,+}. \end{cases} \tag{3.3}$$

Denoting the first product of operators in (3.3) by $O_J^{(1)}$, we see immediately from (3.2) that

$$s_i O_J^{(1)} = O_J^{(1)} s_i. \tag{3.4}$$

For the second product of operators, $O_J^{(2)}$ say, we note that if $\vec{\beta} \in R_{J,+}, \vec{\beta} \neq \vec{\beta}_i$ and $\langle\vec{\beta}, \vec{\beta}_i\rangle \neq 0$, then $s_i\vec{\beta} \in R_{J,+}$ with

$$\langle s_i\vec{\beta}, \vec{\beta}_i\rangle = -\langle\vec{\beta}, \vec{\beta}_i\rangle. \tag{3.5}$$

Thus we can rewrite that product as

$$O_J^{(2)} = \prod_{\substack{\vec{\beta}\in R_{J,+}\\ \vec{\beta}=\vec{e}_{i+1}-\vec{e}_k\ (k>i+1)}} \left(D_{\vec{\beta}} + \frac{1}{\alpha}\right)\left(D_{s_i\vec{\beta}} + \frac{1}{\alpha}\right) \cdot \prod_{\substack{\vec{\beta}\in R_{J,+}\\ \vec{\beta}=\vec{e}_k-\vec{e}_i\ (k<i)}} \left(D_{\vec{\beta}} + \frac{1}{\alpha}\right)\left(D_{s_i\vec{\beta}} + \frac{1}{\alpha}\right)$$

and then use (3.2) and (3.5) to deduce

$$\begin{aligned} s_i \prod_{\substack{\vec{\beta}\in R_{J,+}\\ \vec{\beta}=\vec{e}_{i+1}-\vec{e}_k\ (k>i+1)}} \left(D_{\vec{\beta}} + \frac{1}{\alpha}\right)\left(D_{s_i\vec{\beta}} + \frac{1}{\alpha}\right) &= \prod_{\substack{\vec{\beta}\in R_{J,+}\\ \vec{e}_{i+1}-\vec{e}_k\ (k>i+1)}} \left(D_{s_i\vec{\beta}} + \frac{1}{\alpha} - \frac{s_i}{\alpha}\right)\left(D_{\vec{\beta}} + \frac{1}{\alpha} + \frac{s_i}{\alpha}\right) s_i \\ &= \prod_{\substack{\vec{\beta}\in R_{J,+}\\ \vec{e}_{i+1}-\vec{e}_k\ (k>i+1)}} \left(D_{s_i\vec{\beta}} + \frac{1}{\alpha}\right)\left(D_{\vec{\beta}} + \frac{1}{\alpha}\right) s_i \end{aligned} \tag{3.6}$$

where to obtain the second equality further use has been made of (3.2). Similarly,

$$\begin{aligned} s_i \prod_{\substack{\vec{\beta}\in R_{J,+}\\ \vec{\beta}=\vec{e}_k-\vec{e}_i\ (k<i)}} \left(D_{\vec{\beta}} + \frac{1}{\alpha}\right)\left(D_{s_i\vec{\beta}} + \frac{1}{\alpha}\right) &= \prod_{\substack{\vec{\beta}\in R_{J,+}\\ \vec{e}_k-\vec{e}_i\ (k<i)}} \left(D_{s_i\vec{\beta}} + \frac{1}{\alpha} - \frac{s_i}{\alpha}\right)\left(D_{\vec{\beta}} + \frac{1}{\alpha} + \frac{s_i}{\alpha}\right) s_i \\ &= \prod_{\substack{\vec{\beta}\in R_{J,+}\\ \vec{e}_k-\vec{e}_i\ (k<i)}} \left(D_{s_i\vec{\beta}} + \frac{1}{\alpha}\right)\left(D_{\vec{\beta}} + \frac{1}{\alpha}\right) s_i \end{aligned} \tag{3.7}$$

and so

$$s_i O_J^{(2)} = O_J^{(2)} s_i. \tag{3.8}$$

In the case that $\vec{\beta}_i \in R_{J,+}$, or equivalently $i \in J$, we must also consider the action of s_i on the final factor in (3.3). For this we see from (3.2) and (2.7) that

$$s_i\left(D_{\vec{\beta}_i} + \frac{1}{\alpha}\right) = -D_{\vec{\beta}_i} s_i + \frac{2}{\alpha} + \frac{s_i}{\alpha}$$

and thus for any f such that $s_i f = -f$,

$$s_i\left(D_{\vec{\beta}_i} + \frac{1}{\alpha}\right) f = \left(D_{\vec{\beta}_i} + \frac{1}{\alpha}\right) f. \tag{3.9}$$

An immediate consequence of (3.4), (3.8) and (3.9) is the action of s_i on $O_J S_{\eta^*}^{(I,J)}(z)$, and we can deduce from this that the latter is proportional to $S_{\eta^*}^{(I\cup J,\emptyset)}(z)$.

Proposition 3.1. *For all $i \in I \cup J$,*

$$s_i\big(O_J S_{\eta^*}^{(I,J)}(z)\big) = O_J S_{\eta^*}^{(I,J)}(z). \tag{3.10}$$

Moreover, with

$$c_{\eta^*} := \prod_{\vec{\beta}\in R_{J,+}} \big((\overline{\eta^*}_{\vec{\beta}} + 1)/\alpha\big) \tag{3.11}$$

where $\overline{\eta^*}_\beta := \overline{\eta^*}_j - \overline{\eta^*}_k$ for $\vec{\beta}$ as in (2.5), we have

$$O_J S_{\eta^*}^{(I,J)}(z) = c_{\eta^*} S_{\eta^*}^{(I\cup J,\emptyset)}(z). \tag{3.12}$$

Proof. As already remarked, (3.10) is an immediate consequence of (3.4), (3.8) and (3.9). Also, from (3.1), (2.15) and (2.8)

$$O_J S_{\eta^*}^{(I,J)}(z) = \sum_{\mu\in W_{I\cup J}(\eta^*)} \hat{c}_{\eta^*\mu} c_\mu E_\mu(z) \tag{3.13}$$

where c_μ is specified by (3.11). We know from the remark at the end of the paragraph containing (2.13) that this structural formula together with (3.10) imply (3.12). □

Next we turn our attention to the structure of the image of the product

$$\left(\prod_{\vec{\beta}\in R_{J,+}} (z^{\vec{\beta}} - 1)\right) U_{\eta^*}^{(I,J)}(z)$$

(cf (1.8)) under the action of O_J.

Proposition 3.2. *Let $F(z)$ be an analytic function of $z_1, \ldots, z_N$ in the neighbourhood of $z = 1^N$, and let $\Phi \subseteq R_+$. Suppose $0 \leqslant l \leqslant \#(\Phi)$, then for any $\vec{\beta}_1, \ldots, \vec{\beta}_l$,*

$$D_{\vec{\beta}_1} \cdots D_{\vec{\beta}_l}\left(\prod_{\vec{\beta}\in\Phi}(z^{\vec{\beta}} - 1)\, F(z)\right) = \sum_{\substack{\Omega\subseteq R_+\\ \#(\Omega)=\#(\Phi)-l}} h_\Omega(z) \prod_{\vec{\beta}\in\Omega}(z^{\vec{\beta}} - 1)$$

where $h_\Omega(z)$ is an analytic function in the neighbourhood of $z = 1^N$ dependent on Ω, F, and $\vec{\beta}_1, \ldots, \vec{\beta}_l$.

Proof. It suffices to establish the result for $l = 1$, as the general l result follows by induction. Now, for general f and g and $\vec{\beta} = \vec{e}_j - \vec{e}_k$ $(j < k)$ we have

$$\frac{1}{1 - z^{\vec{\beta}}}(1 - s_{\vec{\beta}})(fg) = \frac{f - s_{\vec{\beta}} f}{1 - z^{\vec{\beta}}} g + \frac{s_{\vec{\beta}} f}{1 - z^{\vec{\beta}}}(1 - s_{\vec{\beta}})g. \tag{3.14}$$

We apply this formula with

$$f = F \prod_{\substack{\vec{\gamma}\in\Phi\\ \langle\vec{\gamma},\vec{\beta}\rangle=0}} (z^{\vec{\gamma}} - 1) \qquad g = \prod_{\substack{\vec{\gamma}\in\Phi\\ \langle\vec{\gamma},\vec{\beta}\rangle\neq0}} (z^{\vec{\gamma}} - 1). \tag{3.15}$$

Since

$$f - s_{\vec{\beta}} f = \prod_{\substack{\vec{\gamma}\in\Phi\\ \langle\vec{\gamma},\vec{\beta}\rangle=0}} (z^{\vec{\gamma}} - 1)(F - s_{\vec{\beta}} F)$$

it follows immediately that the first term has the required structure.

To show that the second term has the required structure, define a total order $<_{\vec{\beta}}$ on $\{\vec{\gamma} \in \Phi | \langle \vec{\gamma}, \vec{\beta} \rangle \neq 0\}$ by the requirement that $\vec{\gamma}$ and $s_{\vec{\beta}}(\vec{\gamma})$ are adjacent in the order. Otherwise the order is arbitrary. In terms of this order we can write

$$(1 - s_{\vec{\beta}}) \prod_{\substack{\vec{\gamma} \in \Phi \\ \langle \vec{\gamma}, \vec{\beta} \rangle \neq 0}} (z^{\vec{\gamma}} - 1) = \sum_{\vec{\alpha} \in \Phi} \prod_{\substack{\vec{\gamma} \in \Phi \\ \vec{\gamma} <_{\vec{\beta}} \vec{\alpha}}} (z^{s_{\vec{\beta}}(\vec{\gamma})} - 1)\big((z^{\vec{\alpha}} - 1) - (z^{s_{\vec{\beta}}(\vec{\alpha})} - 1)\big) \prod_{\substack{\vec{\gamma} \in \Phi \\ \vec{\alpha} <_{\vec{\beta}} \vec{\gamma}}} (z^{\vec{\gamma}} - 1) \tag{3.16}$$

by the telescoping of the sum (the same mechanism responsible for (3.14)). Now for a given $\vec{\alpha}$, if vectors $\vec{\gamma}_1$ and $\vec{\gamma}_2$ satisfy $\vec{\gamma}_1 <_{\vec{\beta}} \vec{\alpha} <_{\vec{\beta}} \vec{\gamma}_2$, then $\vec{\gamma}_1$ and $\vec{\gamma}_2$ are not adjacent so that $s_{\vec{\beta}}(\vec{\gamma}_1) \neq \vec{\gamma}_2$. It follows that all factors in (3.16) are distinct. Because $(z^{\vec{\alpha}} - z^{s_{\vec{\beta}}(\vec{\alpha})})$ is divisible by $(1 - z^{\vec{\beta}})$, we see that the second term in (3.14) with the substitution (3.15) has the sought structure. □

Proposition 3.2 can be used to establish the following structural formula.

Proposition 3.3. *Let $G(z)$ be an analytic function of $z_1, \ldots, z_N$ in the neighbourhood of $z = 1^N$, and let $F(z)$, Φ and the $\vec{\beta}_i$ be as in proposition 3.2. Then for $0 \leqslant l \leqslant \#(\Phi)$, there exist functions $\tilde{h}_\Omega(z)$ analytic in a neighbourhood of $z = 1^N$ such that*

$$D_{\vec{\beta}_1} \cdots D_{\vec{\beta}_l}\bigg(\prod_{\vec{\beta} \in \Phi}(z^{\vec{\beta}} - 1)\, F(z)G(z)\bigg) - G(z)D_{\vec{\beta}_1} \cdots D_{\vec{\beta}_l}\bigg(\prod_{\vec{\beta} \in \Phi}(z^{\vec{\beta}} - 1)\, F(z)\bigg)$$
$$= \sum_{\substack{\Omega \subseteq R_+ \\ \#(\Omega) = \#(\Phi) - l + 1}} \tilde{h}_\Omega(z) \prod_{\vec{\beta} \in \Omega}(z^{\vec{\beta}} - 1). \tag{3.17}$$

Proof. Consider first the case $l = 1$. We can check from the definition (2.2) that

$$D_{\vec{\lambda}}\bigg(\prod_{\vec{\beta} \in \Phi}(z^{\vec{\beta}} - 1)\, F(z)G(z)\bigg)$$
$$= G(z)D_{\vec{\lambda}}\bigg(\prod_{\vec{\beta} \in \Phi}(z^{\vec{\beta}} - 1)\, F(z)\bigg) + \prod_{\vec{\beta} \in \Phi}(z^{\vec{\beta}} - 1)\, F(z)\bigg(\sum_{j=1}^{N} \lambda_j z_j \frac{\partial}{\partial z_j}\bigg)G(z)$$
$$+ \frac{1}{\alpha} \sum_{\vec{\beta} \in R_+} \frac{\langle \vec{\lambda}, \vec{\beta} \rangle}{z^{\vec{\beta}} - 1}\big(G(z) - s_{\vec{\beta}} G(z)\big)s_{\vec{\beta}}\bigg(\prod_{\vec{\gamma} \in \Phi}(z^{\vec{\gamma}} - 1)\, F(z)\bigg). \tag{3.18}$$

The structure (3.17) follows immediately. We now proceed inductively, assuming the result (3.17) for some $1 \leqslant l < \#(\Phi)$, with our task being to then prove its validity for $l \mapsto l+1$. Since $\vec{\beta}_1, \ldots, \vec{\beta}_l$ are arbitrary in the statement of the proposition, then for $\vec{\beta}_1, \vec{\beta}_2, \ldots, \vec{\beta}_{l+1}$,

$$D_{\vec{\beta}_2} \cdots D_{\vec{\beta}_{l+1}}\bigg(\prod_{\vec{\beta} \in \Phi}(z^{\vec{\beta}} - 1)\, F(z)G(z)\bigg) - G(z)D_{\vec{\beta}_2} \cdots D_{\vec{\beta}_{l+1}}\bigg(\prod_{\vec{\beta} \in \Phi}(z^{\vec{\beta}} - 1)\, F(z)\bigg)$$
$$= \sum_{\substack{\Omega \subseteq R_+ \\ \#(\Omega) = \#(\Phi) - l + 1}} \tilde{h}_\Omega(z) \prod_{\vec{\beta} \in \Omega}(z^{\vec{\beta}} - 1) \tag{3.19}$$

for some $\tilde{h}_\Omega(z)$ which are analytic in a neighbourhood of $z = 1^N$. Applying $D_{\vec{\beta}_1}$ to both sides of (3.19) and making use of proposition 3.2 on the right-hand side we see that

$$D_{\vec{\beta}_1} \cdots D_{\vec{\beta}_{l+1}}\bigg(\prod_{\vec{\beta} \in R_\Phi}(z^{\vec{\beta}} - 1)\, F(z)G(z)\bigg) - D_{\vec{\beta}_1}\bigg\{G(z)D_{\vec{\beta}_2} \cdots D_{\vec{\beta}_{l+1}}\bigg(\prod_{\vec{\beta} \in R_\Phi}(z^{\vec{\beta}} - 1)\, F(z)\bigg)\bigg\}$$
$$= \sum_{\substack{\Omega \subseteq R_+ \\ \#(\Omega) = \#(\Phi) - l}} h_\Omega(z) \prod_{\vec{\beta} \in \Omega}(z^{\vec{\beta}} - 1) \tag{3.20}$$

for some $h_\Omega(z)$ which are analytic in a neighbourhood of $z = 1^N$. Now proposition 3.2 also gives that

$$D_{\vec{\beta}_2}\cdots D_{\vec{\beta}_{l+1}}\bigg(\prod_{\vec{\beta}\in\Phi}(z^{\vec{\beta}}-1)\,F(z)\bigg) = \sum_{\substack{\Omega\subseteq R_+\\ \#(\Omega)=\#(\Phi)-l}} \hat{h}_\Omega(z)\prod_{\vec{\beta}\in\Omega}(z^{\vec{\beta}}-1)$$

for some $\hat{h}_\Omega(z)$ which are analytic in a neighbourhood of $z = 1^N$. Application of the already established $l = 1$ case of the present proposition then shows

$$\begin{aligned} D_{\vec{\beta}_1}\bigg\{G(z)D_{\vec{\beta}_2}\cdots D_{\vec{\beta}_{l+1}}\bigg(\prod_{\vec{\beta}\in\Phi}(z^{\vec{\beta}}-1)\,F(z)\bigg)\bigg\} - G(z)D_{\vec{\beta}_1}\cdots D_{\vec{\beta}_{l+1}}\bigg(\prod_{\vec{\beta}\in R_\Phi}(z^{\vec{\beta}}-1)\,F(z)\bigg) \\ = \sum_{\substack{\Omega\subseteq R_+\\ \#(\Omega)=\#(\Phi)-l}} h'_\Omega(z)\prod_{\vec{\beta}\in\Omega}(z^{\vec{\beta}}-1) \qquad (3.21)\end{aligned}$$

for some $h'_\Omega(z)$ which are analytic in a neighbourhood of $z = 1^N$, and this substituted in (3.20) establishes (3.17) in the case $l \mapsto l+1$. □

An immediate corollary of proposition 3.3 is the following evaluation identity.

Corollary 3.4. *In the notation of proposition 3.2, for* $0 \leqslant l \leqslant \#\Phi - 1$,

$$D_{\vec{\beta}_1}\cdots D_{\vec{\beta}_l}\bigg(\prod_{\vec{\beta}\in\Phi}(z^{\vec{\beta}}-1)\,F(z)\bigg)\bigg|_{z=1^N} = 0 \qquad (3.22)$$

while

$$D_{\vec{\beta}_1}\cdots D_{\vec{\beta}_{\#\Phi}}\bigg(\prod_{\vec{\beta}\in\Phi}(z^{\vec{\beta}}-1)\,F(z)\bigg)\bigg|_{z=1^N} = F(z)\bigg|_{z=1^N} D_{\vec{\beta}_1}\cdots D_{\vec{\beta}_{\#\Phi}}\bigg(\prod_{\vec{\beta}\in\Phi}(z^{\vec{\beta}}-1)\bigg)\bigg|_{z=1^N}. \qquad (3.23)$$

In particular

$$O_J S^{(I,J)}_{\eta^*}(z)\bigg|_{z=1^N} = U^{(I,J)}_{\eta^*}(1^N)\bigg(\prod_{\vec{\beta}\in R_{J,+}} D_{\vec{\beta}}\bigg)\bigg(\prod_{\vec{\beta}\in R_{J,+}}(z^{\vec{\beta}}-1)\bigg)\bigg|_{z=1^N}. \qquad (3.24)$$

Proof. The equations (3.22) and (3.23) are immediate consequences of proposition 3.3. Using these equations, (3.24) follows after recalling the definitions (1.8) and (3.1), and noting that

$$\frac{\prod_{\vec{\beta}\in R_{J,+}}(z^{\vec{\beta}}-1)}{\prod_s \Delta_{\tilde{J}_s}(z)} = z^{\vec{\lambda}} \qquad (3.25)$$

for some $\vec{\lambda}$. □

Substituting (3.12), (3.11) and proposition 2.3 in (3.24) we see that

$$U^{(I,J)}_{\eta^*}(1^N) = \frac{|\mathrm{Sym}\, I\cup J|}{\mathcal{M}_{I,\eta^*}}\,\frac{e_{\eta^+}}{d_{\eta^{*(-,-)}}}\,\frac{1}{k_J}\prod_{\vec{\beta}\in R_{J,+}}\big((\overline{\eta^*}_{\vec{\beta}}+1)/\alpha\big) \qquad (3.26)$$

where

$$k_J := \bigg(\prod_{\vec{\beta}\in R_{J,+}} D_{\vec{\beta}}\bigg)\bigg(\prod_{\vec{\beta}\in R_{J,+}}(z^{\vec{\beta}}-1)\bigg)\bigg|_{z=1^N}. \qquad (3.27)$$

With our objective being to compute $U^{(I,J)}_{\eta^*}(1^N)$, we see from (3.26) that the remaining task is to compute k_J as specified by (3.27).

Suppose for a given J we could choose a composition η^* such that $U_{\eta^*}(1^N)$ could be evaluated directly from its definition (1.8). Then because k_J is independent of η^*, that evaluation substituted in (3.26) will specify k_J. To implement this strategy, we consider the particular composition $\eta^* = \delta$ defined by

$$\delta_j = \begin{cases} \max(\tilde{J}_s) - j & j \in \tilde{J}_s \\ 0 & j \notin \tilde{J}. \end{cases} \qquad (3.28)$$

With this definition and the final equality in (2.11) we see that

$$\mathrm{Asym}_J\, z^\delta = \Delta_{\tilde{J}}(z)$$

and thus according to (2.13)

$$S_\delta^{(\emptyset,J)}(z) = \Delta_{\tilde{J}}(z).$$

Substituting this in (1.8) shows

$$U_\delta^{(\emptyset,J)}(1^N) = 1. \qquad (3.29)$$

The value of the quantity k_J can now be deduced by substituting (3.29) in (3.26) and simplifying the result.

Proposition 3.5. *With k_J specified by (3.27) we have*

$$k_J = \prod_s |\tilde{J}_s|!\, \alpha^{-|\delta|} e_\delta. \qquad (3.30)$$

Proof. Substituting (3.28) in (3.26) gives

$$k_J = \prod_s |\tilde{J}_s|!\, \alpha^{-|\delta|} \frac{e_{\delta^+}}{d_{\delta^{(\cdot,-)}}} \prod_{\vec{\beta}\in R_{J,+}} (\bar{\delta}_{\vec{\beta}} + 1)$$

so to obtain (3.28) we must show

$$d_{\delta^{(\cdot,-)}} = \prod_{\vec{\beta}\in R_{J,+}} (\bar{\delta}_{\vec{\beta}} + 1) = \prod_{(i,j)\in\eta^*} (\bar{\delta}_i - \bar{\delta}_{i+j} + 1) \qquad (3.31)$$

where the final equality follows from the definitions of the quantities in the preceding expression. Now after recalling the definition (2.16) of d_η and the meaning of $\delta^{(\cdot,-)}$, we see the left-hand side of (3.31) can be rewritten

$$\prod_s \prod_{i'=0}^{J_s^{\max}-J_s^{\min}} \prod_{j=1}^{i'+1} \big(\alpha(a(J_s^{\min} + i' + 1, j) + 1) + l(J_s^{\min} + i' + 1, j) + 1\big) \qquad (3.32)$$

where the arm and leg lengths are to be calculated with respect to the diagram of $\delta^{(\cdot,-)}$. Similarly, on the right-hand side of (3.31) we can write

$$\prod_s \prod_{i'=0}^{J_s^{\max}-J_s^{\min}} \prod_{j=1}^{i'+1} \big(\bar{\delta}_{J_s^{\max}-i'} - \bar{\delta}_{J_s^{\max}-i'+j} + 1\big). \qquad (3.33)$$

It is easy to check from the definitions that the terms in the products (3.32) and (3.33) are in fact identical, and thus (3.31) does indeed hold. □

Substituting (3.30) in (3.26) and noting $|\mathrm{Sym}\, I \cup J| = (\prod_s |\tilde{I}_s|!)(\prod_s |\tilde{J}_s|!)$ gives the sought evaluation formula.

Proposition 3.6. *We have*

$$U_{\eta^*}^{(I,J)}(1^N) = \frac{\prod_s |\tilde{I}_s|!}{\mathcal{M}_{I,\eta^*}} \frac{e_{\eta^+}}{\alpha^{-|\delta|} e_{\delta^+} d_{\eta^{*(-,-)}}} \prod_{\vec{\beta}\in R_{J,+}} \big((\overline{\eta^*}_{\vec{\beta}} + 1)/\alpha\big). \qquad (3.34)$$

4. Discussion

Let us first compare our result (3.34) with that which can be deduced from the evaluation formula of Dunkl [4]. In our notation, the latter formula is

$$\left.\frac{\mathcal{O}_{I,J}E_\eta(z)}{\prod_s \Delta_{\tilde{J}_s}(z)}\right|_{z=1^N} = \prod_s |\tilde{I}_s|! \frac{e_{\eta^+}}{\alpha^{-|\delta|}e_{\delta^+}} \frac{1}{d_{\eta^*}} \prod_{\vec{\beta}\in R_{J,+}} \left((\overline{\eta^*}_{\vec{\beta}} - 1)/\alpha\right). \tag{4.1}$$

Making use of (2.13) and (2.21) in the case $\eta = \eta^*$, and recalling (1.8), we see that this implies

$$U_{\eta^*}^{(I,J)}(1^N) = \frac{\prod_s |\tilde{I}_s|!}{\mathcal{M}_{I,\eta^*}} \frac{d'_{\eta^{*(-,-)}}}{d'_{\eta^{*(-,0)}} d_{\eta^{*(-,0)}}} \frac{e_{\eta^+}}{\alpha^{-|\delta|}e_{\delta^+}} \prod_{\vec{\beta}\in R_{J,+}} \left((\overline{\eta^*}_{\vec{\beta}} - 1)/\alpha\right). \tag{4.2}$$

Note that this is a different form to the result (3.34). To reconcile the two forms we must have

$$\frac{d'_{\eta^{*(-,-)}} d_{\eta^{*(-,-)}}}{d'_{\eta^{*(-,0)}} d_{\eta^{*(-,0)}}} = \prod_{\vec{\beta}\in R_{J,+}} \frac{\overline{\eta^*}_{\vec{\beta}} + 1}{\overline{\eta^*}_{\vec{\beta}} - 1}. \tag{4.3}$$

In fact (4.3) is just a special case of the following result.

Proposition 4.1. *For $w \in W_J$ we have*

$$\frac{d'_{w\eta^{*(-,0)}} d_{w\eta^{*(-,0)}}}{d'_{\eta^{*(-,0)}} d_{\eta^{*(-,0)}}} = \prod_{\vec{\beta}\in R_{J,+}\cap w^{-1}R_{J,-}} \frac{\overline{\eta^*}_{\vec{\beta}} + 1}{\overline{\eta^*}_{\vec{\beta}} - 1}. \tag{4.4}$$

Proof. This is demonstrated by induction of $l(w)$ (the length of w, i.e. the smallest value of l for which w can be expressed as a product $w = s_{j_1} \dots s_{j_l}$ for $j_1, \dots, j_l \in J$). First, note that for all i, and for all compositions μ such that $s_i\mu \neq \mu$, $\overline{s_i\mu} = s_i\bar{\mu}$. Since $I \cap J = \emptyset$, and I and J satisfy (1.6), then it follows that for all $i \in I$ and all compositions μ, $\overline{s_i\mu}_j = \bar{\mu}_j$ for all $j \in \tilde{J}$. This generalizes to the result that if $w \in W_I$, then for all compositions μ, $\overline{w\mu}_j = \bar{\mu}_j$ for $j \in \tilde{J}$. A consequence of this result is that $\overline{\eta^{*(-,0)}}_j = \overline{\eta^*}_j$ for $j \in \tilde{J}$, and $\overline{\eta^{*(-,0)}}_{\vec{\beta}} = \overline{\eta^*}_{\vec{\beta}}$ for $\vec{\beta} \in R_{J,+}$. Now, (4.4) is clearly true when $w = 1$ (the left-hand side is 1, and the product on the right-hand side is empty, and therefore evaluates to 1). For general $w \in W_J$, $w = s_{j_1} \dots s_{j_l}$ for some $j_1, \dots, j_l \in J$, where $l = l(w)$. It follows that $w^{-1}\vec{\alpha}_{j_1} \in R_{J,-}$, where $\vec{\alpha}_j = \vec{e}_j - \vec{e}_{j+1}$ for $j = 1, \dots, N-1$. Let $w_1 = s_{j_2} \dots s_{j_l}$, then $l(w_1) = l-1$, $w = s_{j_1}w_1$, and $w_1^{-1}\vec{\alpha}_{j_1} \in R_{J,+}$. Since $w_1^{-1}\vec{\alpha}_{j_1} \in R_{J,+}$, then $(w_1\eta^{*(-,0)})_{j_1} > (w_1\eta^{*(-,0)})_{j_1+1}$, and so

$$\frac{d'_{w\eta^{*(-,0)}}}{d'_{w_1\eta^{*(-,0)}}} = \frac{\overline{w_1\eta^{*(-,0)}}_{\vec{\alpha}_{j_1}}}{\overline{w_1\eta^{*(-,0)}}_{\vec{\alpha}_{j_1}} - 1}$$

and

$$\frac{d_{w\eta^{*(-,0)}}}{d_{w_1\eta^{*(-,0)}}} = \frac{\overline{w_1\eta^{*(-,0)}}_{\vec{\alpha}_{j_1}} + 1}{\overline{w_1\eta^{*(-,0)}}_{\vec{\alpha}_{j_1}}}.$$

It follows that

$$\frac{d'_{w\eta^{*(-,0)}} d_{w\eta^{*(-,0)}}}{d'_{w_1\eta^{*(-,0)}} d_{w_1\eta^{*(-,0)}}} = \frac{\overline{w_1\eta^{*(-,0)}}_{\vec{\alpha}_{j_1}} + 1}{\overline{w_1\eta^{*(-,0)}}_{\vec{\alpha}_{j_1}} - 1}.$$

As a consequence of (1.7), then $\overline{w_1\eta^{*(-,0)}} = w_1\overline{\eta^{*(-,0)}}$, so that

$$\frac{d'_{w\eta^{*(-,0)}}d_{w\eta^{*(-,0)}}}{d'_{w_1\eta^{*(-,0)}}d_{w_1\eta^{*(-,0)}}} = \frac{\overline{\eta^{*(-,0)}}_{w_1^{-1}\vec{\alpha}_{j_1}}+1}{\overline{\eta^{*(-,0)}}_{w_1^{-1}\vec{\alpha}_{j_1}}-1} = \frac{\overline{\eta^*}_{w_1^{-1}\vec{\alpha}_{j_1}}+1}{\overline{\eta^*}_{w_1^{-1}\vec{\alpha}_{j_1}}-1}.$$

By the induction hypothesis,

$$\frac{d'_{w_1\eta^{*(-,0)}}d_{w_1\eta^{*(-,0)}}}{d'_{\eta^{*(-,0)}}d_{\eta^{*(-,0)}}} = \prod_{\vec{\beta}\in R_{J,+}\cap w_1^{-1}R_{J,-}} \frac{\overline{\eta^*}_{\vec{\beta}}+1}{\overline{\eta^*}_{\vec{\beta}}-1}$$

since $l(w_1) = l-1$, and so

$$\frac{d'_{w\eta^{*(-,0)}}d_{w\eta^{*(-,0)}}}{d'_{\eta^{*(-,0)}}d_{\eta^{*(-,0)}}} = \frac{\overline{\eta^*}_{w_1^{-1}\vec{\alpha}_{j_1}}+1}{\overline{\eta^*}_{w_1^{-1}\vec{\alpha}_{j_1}}-1}\prod_{\vec{\beta}\in R_{J,+}\cap w_1^{-1}R_{J,-}} \frac{\overline{\eta^*}_{\vec{\beta}}+1}{\overline{\eta^*}_{\vec{\beta}}-1}. \tag{4.5}$$

Suppose $\vec{\beta}\in R_{J,+}\cap w^{-1}R_{J,-}$, then $w\vec{\beta} = s_{j_1}w_1\vec{\beta}\in R_{J,-}$, and so $w_1\vec{\beta}\in R_{J,-}$ or $w_1\vec{\beta} = \vec{\alpha}_{j_1}$. In the first case, $\vec{\beta}\in R_{J,+}\cap w_1^{-1}R_{J,-}$, and in the second, $\vec{\beta} = w_1^{-1}\vec{\alpha}_{j_1}$. Conversely, if $\vec{\beta}\in R_{J,+}\cap w_1^{-1}R_{J,-}$, then $w_1\vec{\beta}\in R_{J,-}$, and $w_1\vec{\beta}\neq -\vec{\alpha}_{j_1}$ since $w_1^{-1}\vec{\alpha}_{j_1}\in R_{J,+}$, and so $w\vec{\beta} = s_{j_1}w_1\vec{\beta}\in R_{J,-}$. Alternatively, if $\vec{\beta} = w_1^{-1}\vec{\alpha}_{j_1}$, then $\vec{\beta}\in R_{J,+}$ and $w\vec{\beta} = s_{j_1}w_1\vec{\beta} = -\vec{\alpha}_{j_1}\in R_{J,-}$. Since $\vec{\beta}\in R_{J,+}\cap w^{-1}R_{J,-}$ is equivalent to $\vec{\beta}\in R_{J,+}\cap w_1^{-1}R_{J,-}$ or $\vec{\beta} = w_1^{-1}\vec{\alpha}_{j_1}$, then (4.5) becomes

$$\frac{d'_{w\eta^{*(-,0)}}d_{w\eta^{*(-,0)}}}{d'_{\eta^{*(-,0)}}d_{\eta^{*(-,0)}}} = \prod_{\vec{\beta}\in R_{J,+}\cap w^{-1}R_{J,-}} \frac{\overline{\eta^*}_{\vec{\beta}}+1}{\overline{\eta^*}_{\vec{\beta}}-1}$$

which is just (4.4) for w. This completes the proof by induction. □

To deduce (4.3) from (4.4), let $w_{0,J}$ be the longest element of W_J, so that $w_{0,J}\eta^{*(-,0)} = \eta^{*(-,-)}$ and $w_{0,J}R_{J,+} = R_{J,-}$. Then, substituting $w = w_{0,J}$ into (4.4) gives

$$\frac{d'_{\eta^{*(-,-)}}d_{\eta^{*(-,-)}}}{d'_{\eta^{*(-,0)}}d_{\eta^{*(-,0)}}} = \prod_{\vec{\beta}\in R_{J,+}\cap w_{0,J}^{-1}R_{J,-}} \frac{\overline{\eta^*}_{\vec{\beta}}+1}{\overline{\eta^*}_{\vec{\beta}}-1} = \prod_{\vec{\beta}\in R_{J,+}} \frac{\overline{\eta^*}_{\vec{\beta}}+1}{\overline{\eta^*}_{\vec{\beta}}-1}$$

as required.

Next we will contrast our strategy to derive (3.34) with that used by Dunkl to derive (4.2). In our method the key ingredient is the operator O_J with its action (3.12) and evaluation property (3.24). In Dunkl's approach the key ingredient is the skew operator ψ_J, constructed so that

$$\psi_J s_j = -s_j\psi_J \qquad j\in J \tag{4.6}$$

and possessing an evaluation property analogous to that of O_J. It follows from (4.6) that for any f such that $s_j f = -f$,

$$s_j(\psi_J f) = \psi_J f. \tag{4.7}$$

The operator O_J has the same action when restricted to this class of f. However, O_J does not exhibit the general algebraic property (4.6). Thus the two approaches are identical in strategy except that in Dunkl's work a more complicated operator ψ_J is constructed which plays the role of our O_J.

The result (4.1), or equivalently (3.34), has been used by Kato and Yamamoto [7] to obtain the expansion of (1.9) in terms of $\{U_{\eta^*}^{(I,J)}\}$ (proposition 4.2 below). This result was then used in the exact computation of the retarded Green function for spin generalizations of the Calogero–Sutherland system (1.1). Here we use the expansion formula to derive a CT identity, which generalizes an integration formula due to Kadell, Kaneko and Macdonald [5, 6, 9]. Let us first present the expansion formula.

Proposition 4.2. *For general r we have*

$$\prod_{i\notin\tilde{J}}(1-x_i)^{r-1}\prod_s\prod_{j\in\tilde{J}_s}(1-x_j)^{r-|\tilde{J}_s|} = \sum_{\eta^*}\frac{\alpha^{|\eta|}}{d'_{\eta^*}}\frac{[1-r]_{\eta^+}^{(\alpha)}}{[1-r]_{\delta^+}^{(\alpha)}}\prod_{\vec{\beta}\in R_{J,+}}\left((\overline{\eta^*}_{\vec{\beta}}-1)/\alpha\right)U_{\eta^*}^{(I,J)}(x). \tag{4.8}$$

Proof. We recall the nonsymmetric Cauchy product expansion

$$\Omega(x,y) := \prod_{i=1}^N\frac{1}{(1-x_iy_i)}\prod_{i,j=1}^N\frac{1}{(1-x_iy_j)^{1/\alpha}} = \sum_\eta\frac{d_\eta}{d'_\eta}E_\eta(x)E_\eta(y) \tag{4.9}$$

(cf (2.22)). Applying the operator $\mathcal{O}_{I,J}$ to the variables x on the right-hand side of (4.9) gives

$$\begin{aligned}\mathcal{O}_{I,J}\Omega(x,y) &= \sum_{\eta^*}\sum_{\mu\in W_{I\cup J}(\eta^*)}\frac{d_\mu}{d'_\mu}a_\mu^{(I,J)}S_{\eta^*}^{(I,J)}(x)E_\mu(y)\\ &= \sum_{\eta^*}a_{\eta^*}^{-1}S_{\eta^*}^{(I,J)}(x)S_{\eta^*}^{(I,J)}(y)\end{aligned} \tag{4.10}$$

where the second equality follows from (2.24), and a_{η^*} is given by (2.26). On the other hand, applying the operator $\mathcal{O}_{I,J}$ directly to the x variables in its definition as a product, and using the Cauchy formula

$$\mathrm{Asym}^{(x)}\prod_{j\in\tilde{J}_s}\frac{1}{(1-x_jy_j)} = \frac{\Delta_{\tilde{J}_s}(x)\Delta_{\tilde{J}_s}(y)}{\prod_{i,j\in\tilde{J}_s}(1-x_iy_j)}$$

we see that

$$\begin{aligned}\mathcal{O}_{I,J}\Omega(x,y) &= \left(\prod_{i\notin\tilde{I}\cup\tilde{J}}\frac{1}{1-x_iy_i}\right)\left(\mathrm{Sym}\prod_{i\in\tilde{I}}\frac{1}{1-x_iy_i}\right)\left(\prod_s\frac{\Delta_{\tilde{J}_s}(x)\Delta_{\tilde{J}_s}(y)}{\prod_{i,j\in\tilde{J}_s}(1-x_iy_j)}\right)\\ &\quad\times\prod_{i,j=1}^N\frac{1}{(1-x_iy_j)^{1/\alpha}}.\end{aligned} \tag{4.11}$$

Equating (4.10) and (4.11), dividing by $\prod_s\Delta_{\tilde{J}_s}(x)\Delta_{\tilde{J}_s}(y)$ and substituting $y=1^N$ we obtain

$$\prod_s|\tilde{I}_s|!\prod_{i\notin\tilde{J}}(1-x_i)^{-1-N/\alpha}\prod_s\prod_{j\in\tilde{J}_s}(1-x_j)^{-|\tilde{J}_s|-N/\alpha} = \sum_{\eta^*}a_{\eta^*}^{-1}U_{\eta^*}^{(I,J)}(x)U_{\eta^*}^{(I,J)}(1^N). \tag{4.12}$$

Now it follows from (2.26) and (4.1) that

$$\frac{U_{\eta^*}^{(I,J)}(1^N)}{a_{\eta^*}\prod_s|\tilde{I}_s|!} = \frac{e_{\eta^+}}{\alpha^{-|\delta|}e_{\delta^+}}\frac{1}{d'_{\eta^*}}\prod_{\vec{\beta}\in R_{J,+}}\left((\overline{\eta^*}_{\vec{\beta}}-1)/\alpha\right). \tag{4.13}$$

The only factor in this expression dependent on the number of variables N is

$$\frac{e_{\eta^+}}{\alpha^{-|\delta|}e_{\delta^+}} = \alpha^{|\eta|}\frac{[1+N/\alpha]_{\eta^+}^{(\alpha)}}{[1+N/\alpha]_{\delta^+}^{(\alpha)}} \tag{4.14}$$

where to obtain the equality use has been made of (2.28). At order $|\eta|$ in x, both sides of (4.12) are thus polynomials in N/α, which are equal for each $N = 1, 2, \ldots$. Thus N/α can be replaced by the continuous variable $-r$ and (4.8) follows. □

Our CT identity will be deduced from (4.8) by making use of the fact that $\{S_{\eta^*}^{(I,J)}(x)\}$ is orthogonal with respect to the inner product

$$\langle f, g\rangle_C := \mathrm{CT}\big(f(x)g(1/x)\big(\Delta(x)\Delta(1/x)\big)^{1/\alpha}\big) \tag{4.15}$$

where $\Delta(x) := \prod_{1\leqslant j<k\leqslant N}(x_k - x_j)$. This CT is well defined for f and g Laurent polynomials and $1/\alpha \in \mathbb{Z}^+$. It is defined beyond these cases as a multidimensional Fourier integral. We require the evaluation of the norm of $S_{\eta^*}^{(I,J)}$ in this inner product [1, 7].

Proposition 4.3. *We have*

$$\frac{\|S_{\eta^*}^{(I,J)}\|_C^2}{\|1\|_C^2} = \frac{|\mathcal{O}_{I,J}|}{\mathcal{M}_{I,\eta^*}}\frac{d'_{\eta^*}d'_{\eta^{*(-,-)}}}{d'_{\eta^{*(-,0)}}d_{\eta^{*(-,0)}}}\frac{e_{\eta^+}}{e'_{\eta^+}} \tag{4.16}$$

where e_{η^+} is as in (2.28) while

$$e'_{\eta^+} = [1 + (N-1)/\alpha]^{(\alpha)}_{\eta^+}. \tag{4.17}$$

Proof. According to (2.15) and the orthogonality of $\{E_\eta\}$ with respect to (4.15) we have

$$\langle S_{\eta^*}^{(I,J)}, E_{\eta^*}\rangle_C = \|E_{\eta^*}\|_C^2 = \frac{d'_{\eta^*}e_{\eta^+}}{d_{\eta^*}e'_{\eta^+}}\|1\|_C^2 \tag{4.18}$$

where the second equality is a formula in [3]. But since the weight function in (4.15) is symmetric

$$\begin{aligned}\langle S_{\eta^*}^{(I,J)}, E_{\eta^*}\rangle_C &= \frac{1}{|\mathcal{O}_{I,J}|}\langle S_{\eta^*}^{(I,J)}, \mathcal{O}_{I,J}E_{\eta^*}\rangle_C = \frac{a_{\eta^*}^{(I,J)}}{|\mathcal{O}_{I,J}|}\|S_{\eta^*}^{(I,J)}\|_C^2\\ &= \frac{\mathcal{M}_{I,\eta^*}}{|\mathcal{O}_{I,J}|}\frac{d'_{\eta^{*(-,0)}}d_{\eta^{*(-,0)}}}{d'_{\eta^*}d'_{\eta^{*(-,-)}}}\|S_{\eta^*}^{(I,J)}\|_C^2.\end{aligned}$$

Equating the two results gives (4.16). □

We are now in a position to present our CT identity.

Proposition 4.4. *We have*

$$\begin{aligned}\mathrm{CT}\Bigg\{&\prod_{i=1}^N(1-x_i)^a\left(1-\frac{1}{x_i}\right)^b\left(\prod_s\prod_{j\in\tilde{J}_s}\left(1-\frac{1}{x_j}\right)^{1-|\tilde{J}_s|}\Delta_{\tilde{J}_s}(1/x)\Delta_{\tilde{J}_s}(x)\right)\\ &\times\big(\Delta(1/x)\Delta(x)\big)^{1/\alpha}U_{\eta^*}(x)\Bigg\}\\ &= \frac{e'_{\eta^+}}{d'_{\eta^*}[-a-b]^{(\alpha)}_{\delta^+}}\prod_{\vec{\beta}\in R_{J,+}}\big((\overline{\eta^*}_{\vec{\beta}}-1)/\alpha\big)\|S_{\eta^*}^{(I,J)}\|_C^2\frac{[-b]^{(\alpha)}_{\eta^+}}{[a+1+(N-1)/\alpha]^{(\alpha)}_{\eta^+}}\\ &\times\prod_{i=1}^N\frac{\Gamma(1+(i-1)/\alpha)}{\Gamma(1+a+(i-1)/\alpha)}\frac{\Gamma(1+a+b+(i-1)/\alpha)}{\Gamma(1+b+(i-1)/\alpha)}.\end{aligned} \tag{4.19}$$

Proof. From the orthogonality of $\{S^{(I,J)}_{\eta^*}\}$ with respect to (4.15) and the definition (1.8) of $U^{(I,J)}_{\eta^*}$, it follows from (4.8) that

$$\mathrm{CT}\left\{\prod_{i\notin\tilde J}\left(1-\frac{1}{x_i}\right)^{r-1}\left(\prod_s\prod_{j\in\tilde J_s}\left(1-\frac{1}{x_j}\right)^{r-|\tilde J_s|}\Delta_{\tilde J_s}(1/x)\Delta_{\tilde J_s}(x)\right)\big(\Delta(1/x)\Delta(x)\big)^{1/\alpha}U_{\eta^*}(x)\right\}$$
$$=\frac{\alpha^{|\eta|}}{d'_{\eta^*}}\frac{[1-r]^{(\alpha)}_{\eta^+}}{[1-r]^{(\alpha)}_{\delta^+}}\prod_{\vec\beta\in R_{J,+}}\big((\overline{\eta^*}_{\vec\beta}-1)/\alpha\big)\|S^{(I,J)}_{\eta^*}\|^2_C. \tag{4.20}$$

Substitute $r=a+b+1$ ($a\in\mathbb{Z}^+$) and make the replacement $\eta^*\mapsto\eta^*+a^N$, where $\eta^*+a^N=(\eta^*_1+a,\dots,\eta^*_N+a)$. Since

$$U_{\eta^*+a^N}(x)=x^aU_{\eta^*}(x) \tag{4.21}$$

we see that

$$\|S^{(I,J)}_{\eta^*+a^N}\|^2_C=\|S^{(I,J)}_{\eta^*}\|^2_C \tag{4.22}$$

while the definition of $\bar\eta_i$ in (2.8) shows $\overline{\eta^*}_{\vec\beta}=\overline{\eta^*}_i-\overline{\eta^*}_j$ for some i, j and thus

$$\overline{\eta^*+a^N}_{\vec\beta}=\overline{\eta^*}_{\vec\beta}. \tag{4.23}$$

The definition (2.16) of d'_{η^*} shows

$$\begin{aligned}d'_{\eta^*+a^N}&=d'_{\eta^*}\prod_{j=1}^{a}\prod_{i=1}^{N}\big(N-i+\alpha(j+\eta^+_i)\big)\\&=d'_{\eta^*}\alpha^{aN}\frac{[a+1+(N-1)/\alpha]^{(\alpha)}_{\eta^+}}{[1+(N-1)/\alpha]^{(\alpha)}_{\eta^+}}\prod_{i=1}^{N}\frac{\Gamma(a+1+(i-1)/\alpha)}{\Gamma(1+(i-1)/\alpha)}\\&=d'_{\eta^*}\alpha^{|\eta|+aN}\frac{[a+1+(N-1)/\alpha]^{(\alpha)}_{\eta^+}}{e'_{\eta^+}}\prod_{i=1}^{N}\frac{\Gamma(a+1+(i-1)/\alpha)}{\Gamma(1+(i-1)/\alpha)}\end{aligned} \tag{4.24}$$

where the second equality follows upon use of (2.28) and the third upon use of (4.17). It also follows from (2.28) that

$$[-a-b]^{(\alpha)}_{\eta^++a^N}=[-b]^{(\alpha)}_{\eta^+}(-1)^{aN}\prod_{j=1}^{N}\frac{\Gamma(1+a+b+(j-1)/\alpha)}{\Gamma(1+b+(j-1)/\alpha)}. \tag{4.25}$$

The formulae (4.21)–(4.25) show that the right-hand side of (4.20) reduces to the right-hand side of (4.19) except that there is an extra factor $(-1)^{aN}$.

On the left-hand side, after noting

$$\prod_{i\notin\tilde J_s}\left(1-\frac{1}{x_i}\right)^{r-1}\prod_s\prod_{j\in\tilde J_s}\left(1-\frac{1}{x_i}\right)^{r-|\tilde J_s|}=\prod_{i=1}^{N}\left(1-\frac{1}{x_i}\right)^{r-1}\prod_s\prod_{j\in\tilde J_s}\left(1-\frac{1}{x_i}\right)^{1-|\tilde J_s|}$$

and making use of (4.21), we obtain the CT on the left-hand side of (4.19), except for an extra factor of $(-1)^{aN}$. □

In the case $I=J=\emptyset$ and thus $\eta^*=\eta$, $S^{(I,J)}_{\eta^*}=E_\eta$, (4.19) gives

$$\frac{\mathrm{CT}\{\prod_{i=1}^N(1-x_i)^a(1-1/x_i)^b(\Delta(1/x)\Delta(x))^{1/\alpha}E_\eta(x)\}}{\mathrm{CT}\{\prod_{i=1}^N(1-x_i)^a(1-1/x_i)^b(\Delta(1/x)\Delta(x))^{1/\alpha}\}}$$
$$=\frac{e'_{\eta^+}}{d'_\eta}\frac{\|E_\eta\|^2_C}{\|1\|^2_C}\frac{[-b]^{(\alpha)}_{\eta^+}}{[a+1+(N-1)/\alpha]^{(\alpha)}_{\eta^+}}=E_\eta(1^N)\frac{[-b]^{(\alpha)}_{\eta^+}}{[a+1+(N-1)/\alpha]^{(\alpha)}_{\eta^+}}$$

where the second equality follows from (4.18) and (2.29). This is the CT identity given in [3]. Summing over appropriate linear combinations of $\eta : |\eta| = \kappa$ so that the nonsymmetric Jack polynomial E_η becomes the symmetric Jack polynomial P_κ gives the original Macdonald–Kadell–Kaneko identity.

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